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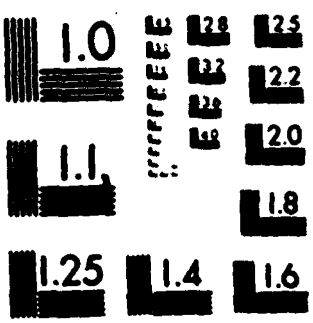
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NAVAL POSTGRADUATE SCHOOL

Monterey, California

THE NORMAL APPROXIMATION AND QUEUE CONTROL
FOR RESPONSE TIMES IN A PROCESSOR-SHARED
COMPUTER SYSTEM MODEL

by

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February 1984

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THE NORMAL APPROXIMATION AND QUEUE CONTROL FOR RESPONSE TIMES IN A PROCESSOR-SHARED COMPUTER SYSTEM MODEL

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ABSTRACT

Represent a time-shared computer system as a group of N terminals, each having submission rate λ and exponential (μ) task durations, with tasks submitted to a central (single) processor. There they are serviced in processor-sharing or time-sliced mode. It is shown that the $R(t)$, the response time conditional on t, the required processing time, becomes approximately normally distributed as t increases. Similar results are derived when N increases.

Variations of the model consider control: an "inside," processor-shared queue services at most c tasks, others queueing first-come first-served "outside." Other possibilities are described and analyzed.

1. Introduction

The abstraction of computer capacity allocation known as processor sharing is an attractive simplification of time slicing, sometimes called round-robin scheduling. The idea is well known (Kleinrock, 1976): given that j jobs or programs are at the execution stage, each receives service equal to one-jth of a time unit per time unit. In other words, if the chance that any single job, processed alone, finishes in $(t,t+h)$ is $\mu h + o(h)$, (exponential-Markov service), then the chance that a particular ("tagged") job in the company of $(j-1)$ others finishes in $(t,t+h)$ is $\mu(h/j) + o(h)$ as $h \to 0$. Processor sharing of the above type tends to be equitable in that it permits short jobs access to processing even if they arrive after, and queue with, longer jobs.

Apparently the first study of delays to arriving and queueing jobs under processor sharing was conducted by Coffman, Muntz, and Trotter (1970). They assumed a steady state M/M/1 system with processor sharing, and were able to determine properties of the response time, R, given the processing time required by the arriving job. Other papers have also appeared.

A recent paper by D. Mitra (1981) analyzes response time, R, under the assumption of a closed system. Idealize the behavior of a system of N terminals and a single computer as a classical machine-repair situation: each thinking terminal (failure-prone machine) applies for computer service at rate λ, and queued or waiting jobs are served at rate μ as long as any jobs are present. Markov assumptions are made throughout, so $X(t)$, the number of jobs at the service stage, is a birth and death process with transition rates

$$\begin{aligned} X(t) = j \rightarrow X(t+h) &= j + 1 : \quad \lambda_j h + o(h) \qquad (1.1) \\ \rightarrow X(t+h) &= j - 1 : \quad \mu_j h + o(h) \\ \rightarrow X(t+h) &= j \qquad : \quad 1 - (\lambda_j+\mu_j)h + o(h) \end{aligned}$$

and in particular $\lambda_j = \lambda(N-j)$, $\mu_j = \mu$ for $j \geq 1$, otherwise being zero. Let processor sharing govern service effort allocation. In Mitra (1981) the distribution of response time is characterized, and the moments (e.g. mean and variance) are found under interesting conditions, such as that the tagged job requiring t time units of processing arrives to find $j - 1$ accompanying jobs; the conditional response time, given only processing requirement t, is given particular attention.

In this paper the previous analysis is generalized and extended. We introduce the idea of processor sharing in an arbitrary birth and death process environment, thus allowing quite general terminal-computer interactions to be represented. In the process, the meaning of "system state at the moment of tagged job arrival" is clarified; see also recent work of Lavenberg and Reiser (1981). Response time characteristics are computed under the assumption that processor-sharing service rates are processor-state-dependent in a more general way than that described earlier; this allows for approximate representation of overhead penalties and also of job scheduling. Other characteristics of tagged job response are also studied, e.g. the accumulated processing work, $W(\tau)$, actually performed on that job by elapsed time $\tau(\tau < t$ = required processing time) following job introduction; note that $W(R) = t$, so the first passage of $W(\tau)$ to t is actually the response time.

Although differential equations may be obtained for transforms of $W(\tau)$ under various initial conditions, and hence, implicitly, for its distribution, the results are far from being explicit and informative. However, central limit theorems for additive functionals of Markov processes, or for cumulative processes, allow the conclusion that the accumulated work accomplished by fixed time τ on a "long" job is approximately normally distributed (Gaussian). This fact in turn allows the conclusion that the response time for a "long" job is also approximately normally distributed. Additionally, the normal approximation may be shown to be valid for our simple model--and probably for others as well--when the number of competing terminals becomes large, i.e. under heavy traffic conditions. The quality of the normal approximations for finite job lengths is currently being assessed by simulation methods.

In the latter part of this paper we describe queue control in a processor-sharing environment. The expedient is to limit the total number of jobs allowed simultaneous processor-shared service at an "inside" queue, with any excess in "first-come, first-served" status in an "outside" queue. Long jobs are also shifted from inside to outside by a sampling mechanism. It is shown that long jobs are favored by a small inside span, c (c being the number simultaneously processor-shared), while short jobs are favored by large c.

2. Mean Response Times

Begin by describing differential equations for the mean response time to be experienced by a tagged, particular, arriving job. Other moments satisfy very similar equations.

(a) Conditioning on Required Time and System State.

Throughout what follows Markovian assumptions are made: service times at the computer are independent and exponential (μ). Generalizations to phase-type distributions are apparently possible.

Let R refer to the response time of a newly arrived job, and

$$m_j(t) = E[R \mid X(0) = j,\ W(R) = t] \ , \tag{2.1}$$

the conditional expectation of response time, given that the job is initially in the company of j others (arrives to find $j - 1$ present) and requires "work" or processing time equal to t . Let $\lambda_j h$ (resp. $\mu_j h$) for small h identify the infinitesimal generator of the accompanying process, so transition rates are as in (1.1).

Consider the possible system changes in $(0,h)$, and subsequently; the following results occur: $\left\{\text{let } \tilde{\mu}_j = \mu(j-1)\frac{r(j)}{j}\right\}$:

$$\begin{aligned} m_j(t) = h + m_j\left(t - \frac{r(j)}{j}\cdot h\right)[1 - (\lambda_j + \tilde{\mu}_j)h] \\ + \lambda_j h m_{j+1}\left(t - \frac{r(j)}{j}\cdot h\right) + \tilde{\mu}_j h m_{j-1}\left(t - \frac{r(j)}{j}\cdot h\right) + o(h) \ . \end{aligned} \tag{2.2}$$

The term $r(j)$ represents the fraction of time the processor actually spends processing when there are j jobs being processed.

Allowing $h \to 0$ one finds the differential equations

$$\frac{r(j)}{j} m_j'(t) = 1 - (\lambda_j + \tilde{\mu}_j)m_j(t) + \lambda_j m_{j+1}(t) + \tilde{\mu}_j m_{j-1}(t) \ . \tag{2.3}$$

This is a standard system of linear differential equations with constant coefficients; initial conditions are $m_j(0) = 0$ for all j .

(b) Conditioning on Required Time.

If one removes the condition that $X(0) = j$ in accordance with the stationary distribution appropriate for an arriving job it follows that the expected response time is linear in the required processing time, t . This holds for quite general birth-and-death process models, and not just for the simple machine-repair setup; see Cohen [1979]. Here is the derivation, in outline.

First, observe that the long-run distribution of $X(0)$, the number of jobs present just after the tagged job enters, is

$$q_j = c\pi_{j-1}\lambda_{j-1} = c\pi_j\mu r(j) \qquad j = 1,2,\ldots,N \,, \tag{2.4}$$

where c is selected so that the q_j's sum to one. Recall that $\pi_j = \pi_0 \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j}$ is the stationary distribution (assumed to exist) of the Markov chain $X(t)$ defined by (1.1) with $\mu_j = \mu r(j)$. This is intuitively apparent, but a formal proof can be based either upon an embedded Markov chain formulation, or upon the theory of additive functionals of a Markov process; see Çinlar ((1975), pp. 269-271). The distribution $\{q_j\}$ has also been given by Kelly, (1979), p. 12.

Use (2.4) to remove the condition that $X(0) = j$; put

$$m(t) = E_{X(0)}E[R|X(0),\ W(R) = t] = \sum_{j=1}^{N} q_j m_j(t) \,. \tag{2.5}$$

Then in terms of the differential equations (2.3); after multiplying through by $j/r(j)$ one obtains $(m_0(t) \equiv 0)$

$$m'(t) = \sum_{j=1}^{N} \frac{1}{r(j)}\, q_j + \sum_{j=1}^{N} q_j \frac{1}{jr(j)} [-(\lambda_j + \tilde{\mu}_j)m_j(t) + \lambda_j m_{j+1}(t) + \tilde{\mu}_j m_{j-1}(t)]$$

$$= \sum_{j=1}^{N} \frac{1}{r(j)}\, q_j \,. \tag{2.6}$$

Thus it follows that the long-run conditional expected response time is linear in the processing time requirement:

$$E[R|W(R) = t] = t\sum_{j=1}^{N} \frac{1}{r(j)}\, q_j = tE\left[\frac{X(0)}{r(X(0))}\right] \,. \tag{2.7}$$

Apparently no such simple form exists for $\mathrm{Var}[R|W(R) = t]$, although Mitra (1981) has given a formula for a particular case. It will be seen, however, that the above variance is indeed proportional to t if t is large.

3. Total Work Completed on a Tagged Job in a Fixed Time

Turn attention now to $W(t)$, the total work expended by the computer on the tagged job by time t after its arrival, given that the tagged job requires exactly t time-units of work for completion. If when the job arrives there are $X(0) = j$ customers present, then

$$W(\tau) = \int_0^{\tau} \frac{r(X(u))}{X(u)}\, du, \qquad X(0) = j \geq 1 . \tag{3.1}$$

(a) The Laplace transform of $W(\tau)$.

Here is the derivation of a differential equation for the Laplace transform of $W(\tau)$:

$$\phi_j(s,\tau;t) = E[e^{-sW(\tau)} | X(0) = j,\ R = t] , \qquad \text{for} \quad j \geq 1 \tag{3.2}$$

Now argue in a manner analogous to the discussion prior to (2.2) to write a backward equation: for $0 < h < t$

$$\begin{aligned} \phi_j(s,\tau;t) = {} & (e^{-sr(j)h/j})[1-(\lambda_j+\tilde{\mu}_j)h]\phi_j(s,\tau-h;t) \\ & + (e^{-sr(j)h/j})\lambda_j h\phi_{j+1}(s,\tau-h;t) \\ & + (e^{-sr(j)h/j})\tilde{\mu}_j h\phi_{j-1}(s,\tau-h;t) + o(h) , \end{aligned} \tag{3.3}$$

leading to

$$\frac{d\phi_j}{d\tau} = -(\lambda_j + \tilde{\mu}_j + sr(j)/j)\phi_j + \lambda_j\phi_{j+1} + \tilde{\mu}_j\phi_{j-1} . \tag{3.4}$$

Initial conditions are

$$\phi_j(s,0;t) = E[e^{-sW(0)} | X(0) = j,\ R = t] = 1 , \qquad t > 0 , \tag{3.5}$$

since initially $W(0) = 0$, regardless of the job requirements or the initial environment.

(b) A central limit theorem for $W(\tau)$.

Examination of (3.1) shows that $W(\tau)$ involves sums of contributions to work accumulated while the system inhabits various states during the period $(0,\tau)$. This suggests that, at least for "long" jobs, i.e. such that required processing time $t \to \infty$, one can anticipate a nearly-Normal distribution for $W(\tau)$. An appropriate central limit theorem that establishes this for finite birth-and-death models can be found in Keilson ((1979), p. 121); call this Theorem K. Alternatively, one can make use of the theory of cumulative processes, see Cox ((1962), pp. 99-101); the latter development is adaptable to models more general than the simple birth-and-death process.

In order to apply Theorem K redefine the infinitesimal generator (1.1) to describe the behavior of the accompaniment, $X'(t)$, of the tagged customer; note that the relevant generator is now

$$X'(t) = j \to X'(t+h) = j+1 : \quad \lambda_j' h + o(h)$$

$$\equiv \lambda_{j+1} h + o(h) \tag{3.6}$$

$$\to X'(t+h) = j-1 : \quad \mu_j' h + o(h)$$

$$\equiv \mu_{j+1}\left(\frac{j}{j+1}\right) h + o(h) \tag{3.7}$$

$$\to X'(t+h) = j \quad : \quad 1-(\lambda_j' + \mu_j')h + o(h) \tag{3.8}$$

for $j = 0,1,2,\ldots, N' = N-1$. Then

$$W(t) = \int_0^t f(X'(u))du \equiv \int_0^t \frac{f(X'(u)+1)du}{X'(u)+1} \tag{3.9}$$

and theorem K states that

$$\frac{W(t) - \zeta t}{\sigma\sqrt{t}} \xrightarrow{(D)} N(0,1) \; ; \tag{3.10}$$

the constants ζ and σ^2 are such that

$$\zeta = \sum_{j=0}^{N'} f(j)\pi_j' \tag{3.11}$$

$$\sigma^2 = 2[f(0), f(1), \ldots, f(N')] \begin{bmatrix} \pi_0' & & 0 \\ & \pi_1' & \\ 0 & & \pi_{N'}' \end{bmatrix} \mathbf{Z} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N') \end{bmatrix} , \tag{3.12}$$

where in the present case the definition of $f(\cdot)$ is implicit in (3.9), and $\mathbf{Z}$ is the matrix

$$\mathbf{Z} = \frac{1}{\gamma}\{[\mathbf{I}-\mathbf{A}+\mathbf{L}]^{-1} - \mathbf{L}\} \tag{3.13}$$

$\mathbf{I}$ being the identity, and

$$\mathbf{\Pi} = \begin{vmatrix} \pi_0' & \pi_1' \cdots \pi_{N'}' \\ \pi_0' & \pi_1' \cdots \pi_{N'}' \\ \pi_0' & \pi_1' \cdots \pi_{N'}' \end{vmatrix} , \tag{3.14}$$

$\mathbf{\Pi}$-rows are steady-state probabilities for the accompaniment, and $\mathbf{A}$ is defined as follows:

$$A_{0,1} = \lambda_0'/\gamma , \quad A_{0,0} = 1 - \lambda_0'/\gamma$$

$$A_{j,j+1} = \lambda_j'/\gamma , \quad A_{j,j-1} = \mu_j'/\gamma , \quad A_{j,j} = 1 - \nu_j/\gamma \tag{3.15}$$

$$\nu_j = \lambda_j' + \mu_j' ; \quad \gamma = \max_j \nu_j .$$

$$(\nu_0 = \lambda_0', \nu_{N'} = \mu_{N'}')$$

(c) A central limit theorem for response time, $R(t)$.

A graph of $W(\tau)$ vs. τ starts with $W(0) = 0$ and increases in random straight-line segments until $W(\tau) = t$. The value of τ at which this occurs, $\tau(t) \equiv R(t)$, is the first-passage time to t of the work process $\{W(\tau), \tau\geq 0\}$, and is the required response time, so

$$P\{W(\tau)<t\} = P\{R(t)>\tau\} . \tag{3.16}$$

Now invoke the previous theorem (3.10) concerning asymptotic normality of $W(\tau)$ and a standard argument of renewal theory, cf. Karlin and Taylor ((1979), pp. 208-209) to see that if $t = \xi\tau + \sqrt{\sigma^2\tau}x$, then, as $t \to \infty$, $\tau \sim \frac{t}{\xi} - \sqrt{\frac{\sigma^2 t}{\xi^3}}\, x$, from which it follows that

$$\frac{R(t) - \alpha t}{\sqrt{\beta^2 t}} \equiv \frac{R(t) - t/\xi}{\sqrt{\frac{\sigma^2}{\xi^3} t}} \overset{(D)}{\to} N(0,1) . \tag{3.17}$$

4. Heavy Traffic Analysis of the Response Time of a Processor-Shared Job

This section investigates the problem of delay of a tagged job requiring t units of processing time when it is accompanied by many others, i.e. is in a heavily loaded system. Restrict attention to the machine repair model in which $\lambda_j = \lambda(N-j)$ and $\mu_j = \mu$, and omit the effect of $r(j)$, i.e. $r(j) \equiv 1$. Let there be N terminals and one processor, with λ^{-1} being the expected terminal think time (exponentially distributed), $\mu = N\mu'$ being the processing rate of arriving jobs; λ and μ' are fixed but N is large and the service rate scaling by N is required in order that queue size be of order N.

Now utilize the fact (Iglehart (1965), and Gaver and Lehoczky (1976)) that if $X(t)$ is the number of jobs at the processing stage at t then $X(t)$ can be approximated by a diffusion process:

$$X(t) = N\,a(t) + \sqrt{N}\,Y(t) \tag{4.1}$$

where $a(t)$ is a deterministic function of time and $\{Y(t)\}$ is, for the present model, a particular Ornstein-Uhlenbeck process. It turns out that when $N \to \infty$

$$\frac{da(t)}{dt} = \lambda(1-a(t)) - \mu' \tag{4.2}$$

or

$$a(\infty) = 1 - \frac{\mu'}{\lambda} \tag{4.3}$$

which is feasible if $\lambda > \mu'$, i.e. under heavy traffic conditions. Furthermore

$$dY(t) = -\lambda Y(t)dt + \sqrt{\lambda(1-a(t)) + \mu'}\,dB(t)\,, \tag{4.4}$$

$\{B(t), t \geq 0\}$ being the standard Wiener process. In the long run,

$$dY(t) = -\lambda Y(t)dt + \sqrt{2\mu'}\,dB(t)\,. \tag{4.5}$$

It is in the environment $X(t)$ described by (4.1) that the tagged job enters. It encounters competition for processor-shared service, and so its accumulated work completed by fixed time τ is essentially

$$W(\tau) = \int_0^\tau \frac{du}{X(u)}\,. \tag{4.6}$$

Apply the approximation and expand to second order terms in N, the number of terminals, to find

$$W(\tau) \simeq \int_0^\tau \frac{du}{Na(u)\left[1 + \frac{Y(u)}{a(u)\sqrt{N}}\right]} \simeq \int_0^\tau \frac{du}{Na(u)} - \int_0^\tau \frac{\sqrt{N}\, Y(u)du}{(Na(u))^2} \tag{4.7}$$

For simplicity, and to enable comparisons with previous results, let $a(v) = a(\infty)$, so the tagged job arrives in the steady state. Expression (4.6) then says that for the approximation advanced here,

$$W(\tau) = \int_0^\tau \frac{du}{E[X(\infty)]} - \frac{\sqrt{N}}{(E[X(\infty)])^2} \int_0^\tau Y(u)du\,, \quad 0 \leq \tau \leq t \tag{4.8}$$

so the expected amount of work done on the tagged job is nearly $\tau/E[X(\infty)]$, and the actual distribution of total work done is approximately Gaussian (integral of an Ornstein-Uhlenbeck process), where the Gaussian property results from the assumption of many accompanying jobs, and not necessarily because the tagged job is long.

Standard calculations applied to (4.7) show that, as $\tau \to \infty$, $E[\int_0^\tau Y(u)du] \simeq 0$ and $Var[\int_0^\tau Y(u)du] \simeq (2\mu'/\lambda^2)\tau$, so the normal approximation to accumulated work $W(\tau)$ has the parameters

$$\xi = \frac{1}{[Na(\infty)]}\,, \qquad \sigma^2 = \frac{2\mu'}{\lambda^2} \cdot \frac{N}{[Na(\infty)]^4} = \frac{2\mu}{\lambda^2[Na(\infty)]^4} \tag{4.9}$$

from which it follows that the parameters of the normal approximation to $R(t)$ are

$$\alpha = \frac{1}{\xi} = [Na(\infty)] \simeq E[X(\infty)]$$

$$\beta^2 = \frac{2\mu}{\lambda^2}\frac{1}{Na(\infty)} \simeq \frac{2\mu}{\lambda^2}\frac{1}{E[X(\infty)]} = \frac{2}{\lambda}\left(\frac{N - E[X(\infty)]}{E[X(\infty)]}\right) . \tag{4.10}$$

These formulas state that if think (demand) rate λ is very large then, since $E[X(\infty)] \to N$, the variance of response time diminishes, while of course expected response time increases like N . This is plausible since in extremely heavy traffic all terminals compete, and the tagged job gets a steady $(1/N)^{\underline{th}}$ of a quantum. For smaller λ the expected response time drops with $E[X(\infty)]$, but response time variance increases.

The above derivations are informative but not rigorous. Semigroup methods of Berman (1979) can be applied to place the results on a mathematically solid basis. Numerical assessment of the results is also of interest.

5. Numerical Comparisons

In this section a brief investigation is reported of the numerical agreement between the very simple formulas from heavy traffic theory for the parameters of the accumulated work distribution and those of direct Markov-chain cumulative process theory origin.

Parameters of Total Work

Examples. Let $\mu = 1$, $N = 25, 50$, with λ varying.

$N = 25$

Rates(λ):	0.01	0.0222...	0.05	0.10	0.15
ξ(M.Ch.)	0.77	0.51	0.16	0.067	.055
(Diffus.)	—	—	0.20	0.067	0.055
σ^2(M.Ch.)	0.44	0.75	0.22	.0045	0.00085
(Diffus.)	—	—	1.28	.0040	0.00079

$N = 50$

Rates(λ):	0.01	0.0222...	0.05	0.10	0.15
ξ(M.Ch.)	0.53	0.13	0.033	0.025	0.023
(Diffus.)	—	0.20	0.033	0.025	0.023
σ^2(M.Ch.)	0.83	0.33	0.0011	0.000081	0.000026
(Diffus.)	—	6.48	0.00099	0.000078	0.000025

The diffusion approximation and Markov chain parameters agree remarkably well when traffic is heavy (large λ), but, as might be feared, diffusion fails miserably for small λ .

6. Queue Control by Service Span and Interruption

In this section we consider queue control. The central processor now has finite service span, c , which may be smaller than the number, N , of terminals. This means that if there are $i \leq c$ jobs in service they are served as before "inside" at a rate $\mu r(i)$, with processor sharing in effect. However, if there are more than c jobs simultaneously requesting service, only c of them are served simultaneously, and at rate $\mu r(c)$, also with processor sharing discipline. The others are queued "outside", with "first-come, first-served" service discipline.

If there are more than c customers requesting service, the customers that are in service "inside" experience independent service interruptions at rate ν. When service is interrupted, each job in service is equally likely to be moved to the end of the queue; thereupon the job at the head of the "outside" queue immediately enters service. Both the imposition of the limited processor sharing, imposed by $c \leq N$, and the interruption process are intended to control queueing by adjusting the relative attention given to short and long jobs.

Markovian assumptions are made throughout, so that $X(t)$, the number of jobs requesting service at time t, is a birth and death process with transition rates given by (1.1).

(a) An Auxiliary Process.

Let R be the response time of a newly arrived tagged job. Since the tagged job may not be served until completion when it first enters service, it is necessary to introduce an auxiliary process $\{Y(t);\ t \geq 0\}$ in order to study R.

In brief summary, if there are i customers (including the tagged job) requesting service at time t and the tagged job is in service, then the state of $Y(t)$ is $(i,0)$. If there are $i > c$ customers requesting service at time t and the tagged job is not in service but is in the $j\underline{th}$ position in queue, the state of $Y(t)$ is (i,j). One can now describe the possible changes in $Y(t)$ in a time interval of length h; details will be presented elsewhere.

Let

$$m_{(i,j)}(t) = E[R \mid Y(0) = (i,j)\ ,\quad W(R) = t]\ , \tag{6.1}$$

the conditional expected response time, given that the tagged job is initially in the company of $(i-1)$ others and either it is being served inside (if $j = 0$) or it is $j\underline{th}$ in the outside queue (if $j > 0$).

Arguments similar to those of Section 2 yield differential equations for $m_{(i,j)}(t)$ which can be solved numerically. A closed-form solution is complicated and uninformative. It is possible to numerically evaluate the mean response time,

$$m(t) = E[R \mid W(R) = t] = \sum_{i,j} q_{(i,j)} m_{(i,j)}(t)\ ,$$

where $q_{(i,j)}$ is the initial distribution encountered by the tagged job.

The mean response time is not generally linear in t for this model. Note that if $c = N$ then this model is equivalent to that considered in Section 2 and hence as shown is Section 2, $m(t)$ is linear in t for that special case.

(b) An Approximation to Expected Response Time.

A useful approximation to the expected response time for a job requiring t units of work is obtained by the following argument. Assume that the service rate for the tagged job is the same throughout its processing and is equal to the rate that it experiences when it first enters the processor. Thus the tagged job requires $t^* = \frac{it}{r(i)}$ units of processing time if it enters when these are $i \leq c$ jobs (including the tagged one) requesting processing. If $i > c$, then $t^* = \frac{ct}{r(c)}$. If $i > c$, then the number of service interruptions during t is Poisson with rate $\frac{\nu}{c}$. Each time service is interrupted, the tagged job spends an expected amount of time $(i-c)\ [\nu + \mu r(c)]^{-1}$ in queue. Thus the expected time spent in queue because of service interuptions is $(i-c)(\frac{\nu}{c})t^*[\nu + \mu r(c)]^{-1}$. If $i > c$, the expected initial wait in queue until the tagged job starts service is $(i-c)[\nu + \mu r(c)]^{-1}$. The resulting approximation to the expected response time is (see (2.4))

$$A = \sum_{i=1}^{c} q_i \frac{it}{r(i)} + \sum_{i=c+1}^{N} q_i \frac{ct}{r(c)} \tag{6.2}$$

$$+ \sum_{i=c+1}^{N} q_i(i-c)[\nu + \mu r(c)]^{-1}\left[\frac{\nu t}{r(c)} + 1\right] .$$

Table 1 gives values for the expected response time, $m(t)$, and the above approximation for various values of λ, μ, ν, c, and t for $r(j) \equiv 1$ and $N = 25$ terminals. The quality of the approximation (6.2) appears to be excellent for all cases considered.

(c) Numerical Implications.

Aspects of the behavior of $m(t)$ to be noted from the table are as follows. If the amount of processing time required, t, is "small", then expected response time is minimized when c is maximized (here $c = 25$); that is, when there is maximal processor sharing and no outside queue. If t is "large", then expected response time is minimized when $c = 1$; that is, when the processor is dedicated solely to the job that is being served, and other jobs queue outside in turn. Note that increasing the rate of service interruptions by changing ν can either increase or decrease the expected response time, depending upon job time requirements.

These behavioral aspects also appear by taking derivatives of the approximate average response time A. In particular,

$$\frac{\partial}{\partial \nu} A \begin{cases} < 0 & \text{if} \quad t < \frac{1}{\mu} , \\ = 0 & \text{if} \quad t = \frac{1}{\mu} , \\ > 0 & \text{if} \quad t > \frac{1}{\mu} . \end{cases}$$

If $r(j) \equiv 1 \quad j = 1,\ldots,N$, then A is decreasing in c if $t < \frac{1}{\mu}$; A is increasing in c if $t > \frac{1}{\mu}$, and A is constant in c if $t = \frac{1}{\mu}$.

Finally, arguments similar to those in Section 3 will show that the response time is approximately normally distributed when the required work is large. Again, details will be provided in later work.

The Expected Response Time

λ	μ	ν	t	Actual	Approx.	C	1	3	5	7	9	11	13	15	17	19	21	23	25
0.1	2	0	0.1	X			2.76	2.04	1.47	1.07	0.82	0.70	0.65	0.64	0.63	0.63	0.63	0.63	0.63
					X		2.76	2.05	1.48	1.07	0.83	0.7	0.65	0.64	—	—	—	—	→
0.1	2	1	0.1	X			2.03	1.56	1.18	0.92	0.76	0.68	0.64	0.63	—	—	—	—	→
					X		2.05	1.57	1.19	.92	.76	.68	.64	.63	—	—	—	—	→
0.1	2	1	0.5	X			3.11	3.12	3.14	3.14	3.15	3.16	—	—	—	—	—	—	→
					X		3.16	—	—	—	—	—	—	—	—	—	—	—	→
1	0.5	2.0	0.4	X			10.72	10.64	10.56	10.48	10.40	10.33	10.25	10.17	10.09	10.01	9.94	9.86	9.8
					X		10.72	10.64	10.56	10.48	10.4	10.33	10.25	10.17	10.09	10.01	9.94	9.86	9.8
.01	2	1	0.1	X			0.15	0.11	—	—	—	—	—	—	—	—	—	—	→
					X		0.15	0.11	—	—	—	—	—	—	—	—	—	—	→

Table 1.

APPENDIX

To show (2.7) and (3.10) yield the same value for $E(\{R|W(\tau) = t\}$ for t large for the case $\mu_j = \mu r(j)$.

First:

$$q(j) = c\pi_j \mu_j$$

$$= c \, \frac{\lambda_0 \times \ldots \times \lambda_{j-1}}{\mu_1 \times \ldots \times \mu_{j-1}} \, .$$

Hence,

$$m \equiv \sum_{j=1}^{\infty} \frac{j}{r(j)} q_j$$

$$= c \sum_{j=1}^{\infty} \frac{j}{r(j)} \frac{\lambda_0 \times \ldots \times \lambda_{j-1}}{\mu_1 \times \ldots \times \mu_{j-1}}$$

Since,

$$\mu_j = \mu r(j)$$

$$m = c\mu \sum_{j=1}^{\infty} j \, \frac{(\lambda_0 \times \ldots \times \lambda_{j-1})}{\mu_1 \times \ldots \times \mu_j}$$

$$= \frac{c\mu}{\prod_{k=1}^{\infty} \mu_k} \sum_{j=1}^{\infty} j(\lambda_0 \times \ldots \times \lambda_{j-1}) \left(\prod_{i=j+1}^{\infty} \mu_i \right)$$

Further,

$$c = \frac{\prod_{k=1}^{\infty} \mu_k}{\sum_{j=1} \lambda_0 \times \ldots \times \lambda_{j-1} \prod_{k=j}^{\infty} \mu_k} \, .$$

Hence,

$$m = \mu \, \frac{\sum_{j=1}^{\infty} j(\lambda_0 \times \ldots \times \lambda_{j-1}) \left(\prod_{k=j+1}^{\infty} \mu_k \right)}{\sum_{j=1}^{\infty} (\lambda_0 \times \ldots \times \lambda_{j-1}) \left(\prod_{k=j}^{\infty} \mu_k \right)} = \frac{\sum_{j=1}^{\infty} j \left(\prod_{i=1}^{j-1} \lambda_i \right) \left(\prod_{k=j+1}^{\infty} \mu_k \right)}{\sum_{j=1}^{\infty} \left(\prod_{i=1}^{j-1} \lambda_i \right) \left(\prod_{k=j}^{\infty} \mu_k \right)}$$

where $\lambda_1 \times \lambda_0 = 1$ by convention.

__Second:__

$$\pi'(j) = \frac{\lambda_1 \times \dots \times \lambda_j}{\mu_2 \times \dots \times \mu_{j+1}} (j+1)\pi'(0) \qquad j = 0,1,\dots \qquad [\lambda_1 \times \lambda_0 \equiv 1]$$

where

$$\pi'(0) = \frac{\prod_{k=2}^{\infty} \mu_k}{\sum_{j=1}^{\infty} j(\lambda_1 \times \dots \times \lambda_{j-1}) \prod_{k=j+1}^{\infty} \mu_k}$$

$$\xi = \sum_{j=0}^{\infty} \frac{r(j+1)}{j+1} \pi'(j) = \sum_{j=1}^{\infty} \frac{r(j)}{j} \pi'(j-1)$$

$$= \sum_{j=1}^{\infty} r(j) \frac{\lambda_1 \times \dots \times \lambda_{j-1}}{\mu_2 \times \dots \times \mu_j} \pi'(0)$$

$$= \frac{1}{\mu} \sum_{j=1}^{\infty} \frac{\lambda_1 \times \dots \times \lambda_{j-1}}{\mu_2 \times \dots \times \mu_{j-1}} \pi'(0)$$

since

$$\mu_{j+1} = \mu r(j+1) \text{ . Thus}$$

$$\xi = \frac{\pi'(0)}{\left[\prod_{k=2}^{\infty} \mu_k\right]} \frac{1}{\mu} \sum_{j=1}^{\infty} \lambda_1 \times \dots \times \lambda_{j-1} \left[\prod_{k=j}^{\infty} \mu_k\right]$$

$$= \frac{\sum_{j=1}^{\infty} (\lambda_1 \times \dots \times \lambda_{j-1}) \left[\prod_{k=j}^{\infty} \mu_k\right]}{\mu\left[\sum_{j=1}^{\infty} j(\lambda_1 \times \dots \times \lambda_{j-1}) \prod_{k=j+1}^{\infty} \mu_k\right]}$$

A comparison of ξ with m shows that

$$m = \frac{1}{\xi} .$$

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